Entanglement and Teleportation in Bipartite System

Satyabrata Adhikari [*]

S. N. Bose National centre for Basic Sciences, Salt lake, Kolkata 700098, India

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Abstract

We present a mathematical formulation of old teleportation protocol (original teleportation protocol introduced by Bennett et.al.) for mixed state and study in detail the role of mixedness of the two qubit quantum channel in a teleportation protocol. We show that maximally entangled mixed state described by the density matrix of rank-4 will be useful as a two qubit teleportation channel to teleport a single qubit mixed state when the teleportation channel parameter $p_1 > \frac{1}{2}$. Also we discuss the case when $p_1 \leq \frac{1}{2}$.

1 Introduction

Bipartite system is nothing but a composite system consisting of only two subsystems A and B situated distant apart. The relation between the state of a composite quantum system as a whole and the configuration of its parts is a very peculiar feature of quantum information theory. It is not obvious that the reduced density operator for system A is in any sense a description for the state of system A. The physical justification for making this identification is that the reduced density operator provides the correct measurement statistics for measurements made on system A [1]. Let us consider the two subsystems A and B described by the Hilbert spaces H_A and H_B respectively, then the composite system is described by the tensor product of the Hilbert spaces H_A and H_B i.e., $H_A \otimes H_B$. If the dimension of the Hilbert spaces H_A and H_B are m and d respectively then the dimension of the composite system is md. Equivalently, if the two subsystems A and B described by the density matrices ρ_A and ρ_B are of order $m \times m$ and $d \times d$ respectively then the composite system is described by the density matrix ρ_{AB} of order $md \times md$. The density matrix ρ_{AB} which describes the bipartite system can also be viewed as $(md \times md)$ ordered partitioned matrix which consists of m^2 (square of the dimension of the Hilbert space H_A) block density matrices each of order $d \times d$. The detection and characterization of entanglement even in the bipartite system is a very difficult and challenging task in the quantum information theory [12, 13, 14, 15, 16, 17].

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[*]satyabrata@bose.res.in

Peres and Horodecki family for the first time showed that inseparability of bipartite composite system can be understood in terms of negative eigenvalues of partial transpose of its density operator [6, 5]. Peres-Horodecki criteria is necessary and sufficient for $2 \otimes 2$ and $2 \otimes 3$ systems but it is only sufficient for higher dimensional cases. Bipartite entangled state can be used in various quantum information processing task like quantum teleportation, superdense coding, quantum cryptography etc. [2, 10, 11]

In this work our discussion will be restricted to only bipartite system. Our main motivation of this work is to investigate whether mixed teleportation channel will be useful to teleport a single qubit mixed state.

The paper is organised as follows. In section-2, we study the properties of semi-definite matrices. In section-3, we study the conventional teleportation scheme for mixed state and show that even mixed state as a teleportation channel will be useful to teleport a single qubit mixed state. In section-4, conclusions are drawn.

2   A Few properties of density matrices in terms of block matrices

Let $\rho_{ij} \in F^{d \times d}$ $\forall$ i,j=0,1,2,...,d-1 denote the block matrices. We define the bipartite composite system in terms of block matrices as

$$[\rho_{AB}] = \begin{pmatrix} [\rho_{00}] & [\rho_{01}] & [\rho_{02}] & .... & [\rho_{0d-1}] \\ [\rho_{01}]^{\dagger} & [\rho_{11}] & [\rho_{12}] & .... & [\rho_{1d-1}] \\ [\rho_{02}]^{\dagger} & [\rho_{12}]^{\dagger} & [\rho_{22}] & .... & [\rho_{2d-1}] \\ .... & .... & .... & .... & .... \\ [\rho_{0d-1}]^{\dagger} & [\rho_{1d-1}]^{\dagger} & [\rho_{2d-1}]^{\dagger} & .... & [\rho_{d-1d-1}] \end{pmatrix} \tag{1}$$

For instance, if we consider $2 \otimes 3$ system then the whole system is described by a 6×6 ordered partition matrix which consist of 3×3 ordered four block matrices. In the matrix form, it is given by

$$[\rho_{AB}] = \begin{pmatrix} [\rho_{00}] & [\rho_{01}] \\ [\rho_{01}^{\dagger}] & [\rho_{11}] \end{pmatrix} \tag{2}$$

where $[\rho_{00}] = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2^* & a_4 & a_5 \\ a_3^* & a_5^* & a_6 \end{pmatrix}$, $[\rho_{01}] = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix}$ and $[\rho_{11}] = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_2^* & b_4 & b_5 \\ b_3^* & b_5^* & b_6 \end{pmatrix}$.

P1. The density matrix $[\rho_{AB}]$ is positive semi-definite iff the diagonal block density matrices $[\rho_{ii}]$ (i=0,1,....d-1) are positive semi-definite.

P2. If the density matrix $[\rho_{AB}]$ is positive-semidefinite then

$$det \begin{pmatrix} det[\rho_{00}] & det[\rho_{01}] & det[\rho_{02}] & .... & det[\rho_{0d-1}] \\ det[\rho_{01}]^{\dagger} & det[\rho_{11}] & det[\rho_{12}] & .... & det[\rho_{1d-1}] \\ det[\rho_{02}]^{\dagger} & det[\rho_{12}]^{\dagger} & det[\rho_{22}] & .... & det[\rho_{2d-1}] \\ .... & .... & .... & .... & .... \\ det[\rho_{0d-1}]^{\dagger} & det[\rho_{1d-1}]^{\dagger} & det[\rho_{2d-1}]^{\dagger} & .... & det[\rho_{d-1d-1}] \end{pmatrix} \leq det[\rho_{AB}] \tag{3}$$

and

$$\begin{pmatrix} tr[\rho_{00}] & tr[\rho_{01}] & tr[\rho_{02}] & .... & tr[\rho_{0d-1}] \\ tr[\rho_{01}]^{\dagger} & tr[\rho_{11}] & tr[\rho_{12}] & .... & tr[\rho_{1d-1}] \\ tr[\rho_{02}]^{\dagger} & tr[\rho_{12}]^{\dagger} & tr[\rho_{22}] & .... & tr[\rho_{2d-1}] \\ .... & .... & .... & .... & .... \\ tr[\rho_{0d-1}]^{\dagger} & tr[\rho_{1d-1}]^{\dagger} & tr[\rho_{2d-1}]^{\dagger} & .... & tr[\rho_{d-1d-1}] \end{pmatrix} \text{ is positive semi-definite} \quad (4)$$

P3. If the density matrix $[\rho_{AB}]$ is positive-semidefinite and represents $2 \otimes d$ system i.e. if $[\rho_{AB}]$ is of the form

$$[\rho_{AB}] = \begin{pmatrix} [\rho_{00}] & [\rho_{01}] \\ [\rho_{01}^{\dagger}] & [\rho_{11}] \end{pmatrix} \tag{5}$$

then

$$tr([\rho_{01}^{\dagger}][\rho_{01}]) \leq \sqrt{tr[\rho_{00}^2]tr[\rho_{11}^2]} \leq tr[\rho_{00}]tr[\rho_{11}] \tag{6}$$

and

$$0 \leq det[\rho_{00}]det[\rho_{11}] - |det[\rho_{01}]|^2 \leq det[\rho_{AB}] \leq det[\rho_{00}]det[\rho_{11}] \tag{7}$$

3 Bennett's et.al. teleportation scheme for mixed states

Quantum teleportation [2] is a fundamental and vital quantum information processing task where an unknown quantum state is transferred from a sender (Alice) to a distant receiver (Bob) using their prior shared entangled state and two bits of classical communication. In the standard teleportation scheme for moving unknown state of a qubit, Alice first performs Bell state measurement on the combined system of input qubit and her half of a maximally entangled pure state. She then uses classical communication channel to tell the measurement result to Bob. After getting the measurement result, Bob applies unitary transformation on his qubit to complete the teleportation protocol. In this section we provide the mathematical formulation (in the light of block matrices) of the conventional teleportation scheme of a qubit by taking the more general two qubit teleportation channel. This formulation will help us in studying the original teleportation protocol in a easier way even with single qubit mixed input state and two qubit mixed teleportation channel. We now proceed step by step to develop our mathematical formulation for teleportation protocol.

Step-I: Let us consider a qubit to be teleported described by the density matrix

$$[\rho_1] = \begin{pmatrix} \rho_{0,0} & \rho_{0,1} \\ \rho_{0,1}^* & \rho_{1,1} \end{pmatrix} \tag{8}$$

To proceed with the teleportation scheme, we need a quantum channel through which Alice sends her message encoding the state (8) to Bob. Thus the two-qubit quantum

channel needed to teleport a single qubit (8) is described by the density matrix

$$[\rho_{23}] = \begin{pmatrix} \rho_{00,00} & \rho_{00,01} & \rho_{00,10} & \rho_{00,11} \\ \rho_{00,01}^* & \rho_{01,01} & \rho_{01,10} & \rho_{01,11} \\ \rho_{00,10}^* & \rho_{01,10}^* & \rho_{10,10} & \rho_{10,11} \\ \rho_{00,11}^* & \rho_{01,11}^* & \rho_{10,11}^* & \rho_{11,11} \end{pmatrix} \tag{9}$$

The qubits 1 and 2 are possessed by Alice and qubit 3 by Bob.

Step-II: Next we combine the single qubit system described by the density operator $[\rho_1]$ and the two-qubit system described by the density operator $[\rho_{23}]$ using tensor product between the two systems. As a result the composite three qubit system is given by

$$[\rho_1] \otimes [\rho_{23}] = \begin{pmatrix} \rho_{000,000} & \rho_{000,001} & \rho_{000,010} & \rho_{000,011} & \rho_{000,100} & \rho_{000,101} & \rho_{000,110} & \rho_{000,111} \\ \rho_{000,001}^* & \rho_{001,001} & \rho_{001,010} & \rho_{001,011} & \rho_{001,100} & \rho_{001,101} & \rho_{001,110} & \rho_{001,111} \\ \rho_{000,010}^* & \rho_{001,010}^* & \rho_{010,010} & \rho_{010,011} & \rho_{010,100} & \rho_{010,101} & \rho_{010,110} & \rho_{010,111} \\ \rho_{000,011}^* & \rho_{001,011}^* & \rho_{010,011}^* & \rho_{011,011} & \rho_{011,100} & \rho_{011,101} & \rho_{011,110} & \rho_{011,111} \\ \rho_{000,100}^* & \rho_{001,100}^* & \rho_{010,100}^* & \rho_{011,100}^* & \rho_{100,100} & \rho_{100,101} & \rho_{100,110} & \rho_{100,111} \\ \rho_{000,101}^* & \rho_{001,101}^* & \rho_{010,101}^* & \rho_{011,101}^* & \rho_{100,101}^* & \rho_{101,101} & \rho_{101,110} & \rho_{101,111} \\ \rho_{000,110}^* & \rho_{001,110}^* & \rho_{010,110}^* & \rho_{011,110}^* & \rho_{100,110}^* & \rho_{101,110}^* & \rho_{110,110} & \rho_{110,111} \\ \rho_{000,111}^* & \rho_{001,111}^* & \rho_{010,111}^* & \rho_{011,111}^* & \rho_{100,111}^* & \rho_{101,111}^* & \rho_{110,111}^* & \rho_{111,111} \end{pmatrix} \tag{10}$$

In the forthcoming step, Alice has to perform Bell-state measurement on her qubits 1 and 2. To do the Bell-state measurement, Alice transforms the basis of the qubits in her side from the computational basis $\{|00>, |01>, |10>, |11>\}$ to the Bell basis $\{|\Phi^+>, |\Phi^->, |\Psi^+>, |\Psi^->\}$, where $|\Phi^\pm> = \frac{1}{\sqrt{2}}(|00> \pm |11>)$ and $|\Psi^\pm> = \frac{1}{\sqrt{2}}(|01> \pm |10>)$. Therefore, when Alice's qubits are written in the Bell basis and Bob's qubit in the computational basis, the eqn. (10) looks like

$$[\rho_1] \otimes [\rho_{23}] = |\Phi^+>_{12}<\Phi^+| \otimes \begin{pmatrix} \rho_{000,000} + \rho_{000,110} & \rho_{000,001} + \rho_{000,111} \\ +\rho_{110,000} + \rho_{110,110} & +\rho_{110,001} + \rho_{110,111} \\ \rho_{000,001}^* + \rho_{000,111}^* & \rho_{001,001} + \rho_{001,111} \\ +\rho_{110,001}^* + \rho_{110,111}^* & +\rho_{111,001} + \rho_{111,111} \end{pmatrix}$$

$$+ |\Phi^->_{12}<\Phi^-| \otimes \begin{pmatrix} \rho_{000,000} - \rho_{000,110} & \rho_{000,001} - \rho_{000,111} \\ -\rho_{110,000} + \rho_{110,110} & -\rho_{110,001} + \rho_{110,111} \\ \rho_{000,001}^* - \rho_{000,111}^* & \rho_{001,001} - \rho_{001,111} \\ -\rho_{110,001}^* + \rho_{110,111}^* & -\rho_{111,001} + \rho_{111,111} \end{pmatrix}$$

$$+ |\Psi^+>_{12}<\Psi^+| \otimes \begin{pmatrix} \rho_{010,010} + \rho_{010,100} & \rho_{010,011} + \rho_{010,101} \\ +\rho_{100,010} + \rho_{100,100} & +\rho_{100,011} + \rho_{100,101} \\ \rho_{010,011}^* + \rho_{010,101}^* & \rho_{011,011} + \rho_{011,101} \\ \rho_{100,011}^* + \rho_{100,101}^* & \rho_{101,011} + \rho_{111,111} \end{pmatrix}$$

$$+ |\Psi^->_{12}<\Psi^-| \otimes \begin{pmatrix} \rho_{010,010} + \rho_{010,100} & \rho_{010,011} - \rho_{010,101} \\ +\rho_{100,010} + \rho_{100,100} & -\rho_{100,011} + \rho_{100,101} \\ \rho_{010,011}^* - \rho_{010,101}^* & \rho_{011,011} - \rho_{011,101} \\ -\rho_{100,011}^* + \rho_{100,101}^* & -\rho_{101,011} + \rho_{111,111} \end{pmatrix}$$

$$+ \text{ other terms containing } |\Psi^+>_{12}< \Psi^-|, \quad |\Psi^->_{12}< \Psi^+| \text{ etc.} \tag{11}$$

Step-III: Alice performs Bell state measurement on her qubit and sends the measurement result to Bob by expending two classical bits.

Step-IV: After getting the measurement result, Bob operates with some unitary operator on the received state to retrieve the state sent by Alice.

Illustration

Let us understand the above steps by taking a specific example:

Suppose an arbitrary input state that Alice wants to teleport is described by the density matrix

$$[\rho_1] = \begin{pmatrix} x & y \\ y^* & 1-x \end{pmatrix} \tag{12}$$

Here the input parameters x and y are unknown to Alice.

Let the two-qubit teleportation channel shared by two distant partners Alice and Bob be given by

$$[\rho_{23}] = \begin{pmatrix} a & 0 & 0 & e \\ 0 & b & c & 0 \\ 0 & c^* & d & 0 \\ e^* & 0 & 0 & 1-a-b-d \end{pmatrix} \tag{13}$$

The composite three qubit system $[\rho_1] \otimes [\rho_{23}]$ can be written as

$$\begin{pmatrix}
ax & 0 & 0 & xe & ya & 0 & 0 & ye \\
0 & xb & xc & 0 & 0 & yb & yc & 0 \\
0 & c^*x & dx & 0 & 0 & yc^* & yd & 0 \\
xe^* & 0 & 0 & x(1-a-b-d) & ye^* & 0 & 0 & y(1-a-b-d) \\
ay^* & 0 & 0 & ey^* & (1-x)a & 0 & 0 & e(1-x) \\
0 & by^* & cy^* & 0 & 0 & (1-x)b & (1-x)c & 0 \\
0 & y^*c^* & dy^* & 0 & 0 & (1-x)c^* & (1-x)d & 0 \\
y^*e^* & 0 & 0 & (1-a-b-d)y^* & (1-x)e^* & 0 & 0 & (1-x)(1-a-b-d)
\end{pmatrix} \tag{14}$$

Using eqn.(11), eqn. (14) can be rewritten as

$$[\rho_1] \otimes [\rho_{23}] = |\Phi^+>_{12}< \Phi^+| \otimes \begin{pmatrix} xa+(1-x)d & y^*c^*+ye \\ yc+y^*e^* & xb+(1-x)(1-a-b-d) \end{pmatrix}$$

$$+|\Phi^->_{12}< \Phi^-| \otimes \begin{pmatrix} xa+(1-x)d & -y^*c^*-ye \\ -yc-y^*e^* & xb+(1-x)(1-a-b-d) \end{pmatrix}$$

$$+|\Psi^+>_{12}< \Psi^+| \otimes \begin{pmatrix} xd+(1-x)a & yc^*+y^*e \\ cy^*+ye^* & x(1-a-b-d)+(1-x)b \end{pmatrix}$$

$$+|\Psi^->_{12}< \Psi^-| \otimes \begin{pmatrix} xd+(1-x)a & -yc^*-y^*e \\ -cy^*-ye^* & x(1-a-b-d)+(1-x)b \end{pmatrix}$$

$$+ \text{ other terms containing } |\Psi^+>_{12}< \Psi^-|, \quad |\Psi^->_{12}< \Psi^+| \text{ etc.} \tag{15}$$

Thereafter Alice performs Bell state measurement on her qubit and sends the measurement result to Bob.

If Alice's measurement outcome is $|\Phi^+>$ or $|\Phi^->$ or $|\Psi^+>$ or $|\Psi^->$ then the corresponding states received by Bob are described by the density matrices

$$[\rho_B^{|\Phi^\pm>}] = (1/N) \begin{pmatrix} xa + (1-x)d & \pm(y^*c^* + ye) \\ \pm(yc + y^*e^*) & xb + (1-x)(1-a-b-d) \end{pmatrix}$$

$$\text{or} \quad [\rho_B^{|\Psi^\pm>}] = (1/N_1) \begin{pmatrix} xd + (1-x)a & \pm(yc^* + y^*e) \\ \pm(cy^* + ye^*) & x(1-a-b-d) + (1-x)b \end{pmatrix} \quad (16)$$

where $N = x(a+b) + (1-x)(1-a-b)$ and $N_1 = x(1-a-b) + (1-x)(a+b)$ are the normalization constants.

Now before going to fourth step of the teleportation protocol, we want to simplify the situation and consider the maximally entangled mixed state (MEMS) [3, 4] as a two qubit teleportation channel shared by Alice and Bob. Therefore the two qubit teleportation channel (13) reduces to

$$[\rho_{23}]^{MEMS} = \begin{pmatrix} \frac{p_1+p_3}{2} & 0 & 0 & \frac{p_3-p_1}{2} \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_4 & 0 \\ \frac{p_3-p_1}{2} & 0 & 0 & 1 - p_2 - p_4 - \frac{p_1+p_3}{2} \end{pmatrix} \quad (17)$$

where $p_1 \geq p_2 \geq p_3 \geq p_4$ and $p_1 + p_2 + p_3 + p_4 = 1$.

CASE-I: When $rank([\rho_{23}]^{MEMS}) = 1$ i.e., when $p_2 = p_3 = p_4 = 0$.
In this case $[\rho_{23}]^{MEMS}$ reduces to maximally entangled pure state (MEPS). Therefore,

$$[\rho_{23}]^{MEPS} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad (18)$$

The maximum concurrence [7] of the state described by the density matrix $[\rho_{23}]^{MEPS}$ is unity.

If Alice uses $[\rho_{23}]^{MEPS}$ as a teleportation channel to teleport an unknown qubit described by the density matrix (12) then the state after Alice's measurement is given by

$$[\rho_{B_1}^{|\Phi^\pm>}] = \begin{pmatrix} x & \mp y \\ \mp y^* & 1-x \end{pmatrix} \quad \text{and} \quad [\rho_{B_2}^{|\Psi^\pm>}] = \begin{pmatrix} 1-x & \mp y^* \\ \mp y & x \end{pmatrix} \quad (19)$$

According to the measurement results $|\Phi^+>$, $|\Phi^->$, $|\Psi^+>$ or $|\Psi^->$ sent by Alice with the help of two classical bits, Bob operates with unitary operators σ_z, I, σ_y and σ_x on his received qubit to retrieve the original state undistorted.

CASE-II: When $rank([\rho_{23}]^{MEMS}) = 2$ i.e., when $p_3 = p_4 = 0$. In this case $[\rho_{23}]^{MEMS}$ reduces to

$$[\rho_{23}]^{MEMS}_{r2} = \begin{pmatrix} \frac{p_1}{2} & 0 & 0 & \frac{-p_1}{2} \\ 0 & 1-p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-p_1}{2} & 0 & 0 & \frac{p_1}{2} \end{pmatrix} \tag{20}$$

The maximum concurrence of the state $[\rho_{23}]^{MEMS}_{r2}$ is $C([\rho_{23}]^{MEMS}_{r2}) = p_1$.

If Alice uses $[\rho_{23}]^{MEMS}_{r2}$ as a teleportation channel to teleport an unknown qubit described by the density matrix (12) then the state received by Bob after Alice's measurement is given by

$$[\rho_{B_3}^{|\Phi^\pm>}] = \frac{1}{N} \begin{pmatrix} \frac{xp_1}{2} & \mp\frac{yp_1}{2} \\ \mp\frac{y^*p_1}{2} & x(1-p_1) + \frac{(1-x)p_1}{2} \end{pmatrix} \quad \text{and}$$

$$[\rho_{B_4}^{|\Psi^\pm>}] = \frac{1}{N_1} \begin{pmatrix} \frac{(1-x)p_1}{2} & \mp\frac{y^*p_1}{2} \\ \mp\frac{yp_1}{2} & \frac{xp_1}{2} + (1-x)(1-p_1) \end{pmatrix} \tag{21}$$

where $N = x(1 - \frac{p_1}{2}) + (1-x)\frac{p_1}{2}$ and $N_1 = x\frac{p_1}{2} + (1-x)(1-\frac{p_1}{2})$.

Based on measurement results $|\Phi^+>$, $|\Phi^->$, $|\Psi^+>$ or $|\Psi^->$, Bob operates with unitary operators σ_z, I, σ_y and σ_x respectively on his received qubit and therefore the state given in (21) reduces to

$$\sigma_z[\rho_B^{|\Phi^+>}]\sigma_z^\dagger = I[\rho_B^{|\Phi^->}]I \equiv [\rho_3^{B_3}] = \frac{1}{N} \begin{pmatrix} \frac{xp_1}{2} & \frac{yp_1}{2} \\ \frac{y^*p_1}{2} & x(1-p_1) + \frac{(1-x)p_1}{2} \end{pmatrix} \quad \text{and}$$

$$\sigma_y[\rho_B^{|\Psi^+>}]\sigma_y^\dagger = \sigma_x[\rho_B^{|\Psi^->}]\sigma_x^\dagger \equiv [\rho_3^{B_4}] = \frac{1}{N_1} \begin{pmatrix} \frac{xp_1}{2} + (1-x)(1-p_1) & \frac{yp_1}{2} \\ \frac{y^*p_1}{2} & \frac{(1-x)p_1}{2} \end{pmatrix} \tag{22}$$

From (22) it is clear that even after unitary transformation Bob could not retrieve the sent state quite properly. Thus to see how good the input state is teleported we have to calculate the square of the Hilbert Schmidt norm [18] of the difference between two density matrices $[\rho_1]$ at Alice's side and $[\rho_3^{B_3}]$ or $[\rho_3^{B_4}]$ (corresponding to different measurements) at Bob's side.

The Hilbert Schmidt norm is given by

$$D_1 = Tr([\rho_1] - [\rho_3^{B_3}])^2 = \frac{2}{N^2}(x^2(1-p_1)^2(x^2 + |y|^2)) \quad \text{and}$$

$$D_2 = Tr([\rho_1] - [\rho_3^{B_4}])^2 = \frac{2}{N_1^2}((1-x)^2(1-p_1)^2((1-x)^2 + |y|^2)) \tag{23}$$

Here we note that if we put $p_1 = 1$ in (23) then we obtain the result as in case-I (i.e., rank-1 case) where the state (12) is perfectly teleported (i.e., without introducing any error) via quantum channel (18).

Let us consider a class of input states given by

$$[\rho_1^{in}] = \begin{pmatrix} \frac{1}{2} & y \\ y^* & \frac{1}{2} \end{pmatrix} \tag{24}$$

Here also the input parameter y is completely unknown to the sender.

To quantify the mixedness of the state (24), we calculate the linear entropy S_L [19]

$$S_L = \frac{4}{3}(1 - Tr([\rho_1^{in}])^2) = \frac{8}{3}(\frac{1}{4} - |y|^2) \tag{25}$$

We note here that for lower values (in the neighborhood of zero) of $|y|$ the input qubit (24) is maximally mixed. As the values of $|y|$ proceed towards $\frac{1}{2}$ the mixedness of the input qubit decreases.

For the above class of input states we find that the Hilbert Schmidt norms D_1 and D_2 are equal and given by

$$D_{12} = 2(1 - p_1)^2(\frac{1}{4} + |y|^2), \quad 0 \leq |y| \leq \frac{1}{2} \tag{26}$$

CASE-III: When $rank([\rho_{23}]^{MEMS}) = 3$ i.e., when $p_4 = 0$. Without any loss of generality, we assume $p_1 = p_2$. Therefore the teleportation channel $[\rho_{23}]^{MEMS}$ reduces to

$$[\rho_{23}]_{r3}^{MEMS} = \begin{pmatrix} \frac{1-p_1}{2} & 0 & 0 & \frac{1-3p_1}{2} \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1-3p_1}{2} & 0 & 0 & \frac{1-p_1}{2} \end{pmatrix} \tag{27}$$

The maximum concurrence of the state $[\rho_{23}]_{r3}^{MEMS}$ is $C([\rho_{23}]_{r3}^{MEMS}) = 3p_1 - 1$.

If Alice uses $[\rho_{23}]_{r3}^{MEMS}$ as a teleportation channel to teleport an unknown qubit described by the density matrix (12) then the state after Alice's measurement and Bob's unitary transformation (like in case-II) is given by

$$[\rho_3^{B_5}] = \frac{1}{N} \begin{pmatrix} \frac{x(1-p_1)}{2} & \frac{y(3p_1-1)}{2} \\ \frac{y^*(3p_1-1)}{2} & xp_1 + \frac{(1-x)(1-p_1)}{2} \end{pmatrix} \text{ and}$$

$$[\rho_3^{B_6}] = \frac{1}{N_1} \begin{pmatrix} \frac{x(1-p_1)}{2} + (1-x)p_1 & \frac{y(3p_1-1)}{2} \\ \frac{y^*(3p_1-1)}{2} & \frac{(1-x)(1-p_1)}{2} \end{pmatrix} \tag{28}$$

where $N = x(\frac{1+p_1}{2}) + (1-x)(\frac{1-p_1}{2})$ and $N_1 = x(\frac{1-p_1}{2}) + (1-x)(\frac{1+p_1}{2})$.

The Hilbert Schmidt norm of the difference between two density matrices $[\rho_1]$ at Alice's side and $[\rho_3^{B_5}]$ or $[\rho_3^{B_6}]$ (corresponding to different measurements) at Bob's side is given by

$$D_3 = Tr([\rho_1] - [\rho_3^{B_5}])^2 = \frac{2}{N^2}(x^4 p_1^2 + |y|^2(1 - 2p_1 + p_1 x)^2) \quad \text{and}$$

$$D_4 = Tr([\rho_1] - [\rho_3^{B_6}])^2 = \frac{2}{N_1^2}((1-x)^4 p_1^2 + |y|^2(1 - p_1 - xp_1)^2) \tag{29}$$

Interestingly we find that for the same class of input state (24), the Hilbert Schmidt norms D_3 and D_4 are equal and given by

$$D_{34} = 4(\frac{p_1^2}{8} + 2|y|^2(1 - \frac{3p_1}{2})^2), \quad 0 \leq |y| \leq \frac{1}{2} \tag{30}$$

CASE-IV: When $rank([\rho_{23}]^{MEMS}) = 4$ i.e., when $p_4 \neq 0$. Without any loss of generality, we assume $p_2 = p_3 = p_4$. Therefore the teleportation channel $[\rho_{23}]^{MEMS}$ reduces to

$$[\rho_{23}]^{MEMS}_{r4} = \begin{pmatrix} \frac{1+2p_1}{6} & 0 & 0 & \frac{1-4p_1}{6} \\ 0 & \frac{1-p_1}{3} & 0 & 0 \\ 0 & 0 & \frac{1-p_1}{3} & 0 \\ \frac{1-4p_1}{6} & 0 & 0 & \frac{1+2p_1}{6} \end{pmatrix} \tag{31}$$

The maximum concurrence of the state $[\rho_{23}]^{MEMS}_{r4}$ is $C([\rho_{23}]^{MEMS}_{r4}) = 2p_1 - 1$.
If Alice uses $[\rho_{23}]^{MEMS}_{r4}$ as a teleportation channel to teleport an unknown qubit described by the density matrix (12) then interestingly we find that corresponding to each of Alice's measurement outcome $(|\Phi^+>, |\Phi^->, |\Psi^+>, |\Psi^->)$ and Bob's unitary operation $(\sigma_z, I, \sigma_y, \sigma_x)$, the state in Bob's side reduces to

$$[\rho_3^{B7}] = \begin{pmatrix} \frac{x(2p_1+1)}{3} + \frac{2(1-x)(1-p_1)}{3} & \frac{y(1-4p_1)}{3} \\ \frac{y^*(4p_1-1)}{3} & \frac{2x(1-p_1)}{3} + \frac{(1-x)(1+2p_1)}{3} \end{pmatrix} \tag{32}$$

In this case the Hilbert Schmidt norm is given by

$$D_5 = Tr([\rho_1] - [\rho_3^{B7}])^2 = \frac{8(1-p_1)^2((2x-1)^2 + 4|y|^2)}{9} \tag{33}$$

For the class of input states given in (24), the eqn. (33) reduces to

$$D_{56} = \frac{32(1-p_1)^2|y|^2}{9}, \quad 0 \leq |y| \leq \frac{1}{2} \tag{34}$$

It is to be noted that the family of Werner states of the form

$$[\rho_{23}]^W = \begin{pmatrix} \frac{1+r}{4} & 0 & 0 & \frac{-r}{2} \\ 0 & \frac{1-r}{4} & 0 & 0 \\ 0 & 0 & \frac{1-r}{4} & 0 \\ \frac{-r}{2} & 0 & 0 & \frac{1+r}{4} \end{pmatrix} \tag{35}$$

falls under this category because the rank of $[\rho_{23}]^W$ is also 4.
Comparing the density matrix $[\rho_{23}]^{MEMS}_{r4}$ with $[\rho_{23}]^W$, we find the relationship between the parameters r and p_1 as

$$p_1 = \frac{1 + 3r}{4} \tag{36}$$

In this case, the distortion D_{56} given in (34) reduces to

$$D_{56}^W = 2(1-r)^2|y|^2 \tag{37}$$

Since the distortion D_{56}^W depends on the input state parameter so averaging over all input states, we get

$$\overline{D}_{56}^W = \frac{(1-r)^2}{12} \tag{38}$$

Now we can speculate few cases:

(i) If $r = 1$, then the teleportation channel would be maximally entangled pure state and as expected, the distortion of the output state from the input state is zero.

(ii) If $r = 0$, then the teleportation channel would be maximally mixed state and hence the average distortion is maximum in this case and it is found out to be $\frac{1}{12}$.

(iii) If $0 < r < 1$ then the quantum teleportation channel would be any channel lying between maximally entangled pure state and maximally mixed state. In this case, the average distortion is lying between 0 and $\frac{1}{12}$.

3.1 Graphical study of the teleportation scheme

In this section we analyze (i) graph for Hilbert Schmidt distance D against input state parameter $|y|$ for different values of teleportation channel parameter p_1 and (ii) graph for linear entropy S_L versus teleportation channel parameter p_1.

We make some interesting observations from the above graphs:

G1. The Hilbert Schmidt distances D_{12}, D_{34} and D_{56} are increasing function of $|y|^2$ for any given p_1.

G2-I. When $0 \le p_1 \le \frac{1}{2}$, the ordering of the Hilbert Schmidt distances D_{12} and D_{34} is given by (i) $D_{34} \le D_{12}$, if $0 \le |y|^2 \le \frac{1-2p_1}{4(8p_1^2-4p_1+3)}$ and (ii) $D_{12} < D_{34}$, if $\frac{1-2p_1}{4(8p_1^2-4p_1+3)} < |y|^2 \le \frac{1}{4}$. The distances D_{12} and D_{34} are equal only when $|y|^2 = \frac{1-2p_1}{4(8p_1^2-4p_1+3)}$. In the graph the intersection point of the curves for D_{12} and D_{34} is shown at P.

G2-II. When $\frac{1}{2} < p_1 < 1$, the ordering of D_{12}, D_{34} and D_{56} is given by $D_{56} < D_{12} < D_{34}$. This observation provide us a clue to arrive at a conclusion that the two qubit state described by the density matrices of rank-4 will be the better option to opt as a teleportation channel when the channel parameter p_1 is greater than $\frac{1}{2}$.

G3-I. As we increase the value of p_1, the minimum value of D_{12} is attained at $|y| = 0$ and this minimum value is shifted downward along vertical axis. This shows that the teleportation of the single qubit mixed state will be better for higher values of p_1 (i.e., for those values of p_1 which lies in the neighborhood of unity) via the two qubit state described by rank-2 density matrices as a teleportation channel.

G3-II. As we increase the value of p_1 starting from $\frac{1}{3}$, the minimum value of D_{34} is also attained at $|y| = 0$ but this minimum value is shifted upward along vertical axis. This means that the teleportation of the single qubit mixed state will be worse for higher values of p_1 when we use the two qubit state described by rank-3 density matrices as a teleportation channel.

G3-III. The minimum value of D_{56} remain fixed at $|y| = 0$ for $p_1 > \frac{1}{2}$.

G4. Let $S_L^{r_2}$, $S_L^{r_3}$ and $S_L^{r_4}$ denote the linear entropies of two qubit teleportation channel described by the density matrix of rank-2, rank-3 and rank-4 respectively.
(i) When $p_1 < \frac{1}{2}$, the ordering of the linear entropies is given by $S_L^{r_2} < S_L^{r_3} < S_L^{r_4}$.
(ii)When $p_1 > \frac{1}{2}$, the ordering of the linear entropies is given by $S_L^{r_3} < S_L^{r_2} < S_L^{r_4}$.
We note that the curve for the linear entropies $S_L^{r_2}$ and $S_L^{r_3}$ intersect at R $(0.5, 0.66)$.
It is therefore clear that the mixedness of the two qubit teleportation channel described by the density matrix of rank-4 is greater than the mixedness of the density matrix of rank-2 and rank-3 respectively for any value of the parameter p_1.

4   Conclusion

We have studied partial trace and partial transposition in terms of block matrices and thereafter derived an entanglement condition for bipartite $2 \otimes 2$ system. Also we have presented a mathematical formulation of the original teleportation scheme. In the teleportation scheme, we consider a particular set of single qubit mixed state which is to be teleported. Then we find that the success of the teleportation depends on the channel parameter p_1 i.e. when $p_1 < \frac{1}{2}$, the two qubit teleportation channel described by the density matrices of rank-2 or rank-3 will be useful to teleport a single qubit mixed state described by the parameter $\frac{1-2p_1}{4(8p_1^2-4p_1+3)} < |y|^2 \leq \frac{1}{4}$ or $0 \leq |y|^2 \leq \frac{1-2p_1}{4(8p_1^2-4p_1+3)}$. When $p_1 > \frac{1}{2}$, the maximally entangled mixed two qubit teleportation channel described by the density matrix of rank-4 should be used to teleport a single qubit mixed state.

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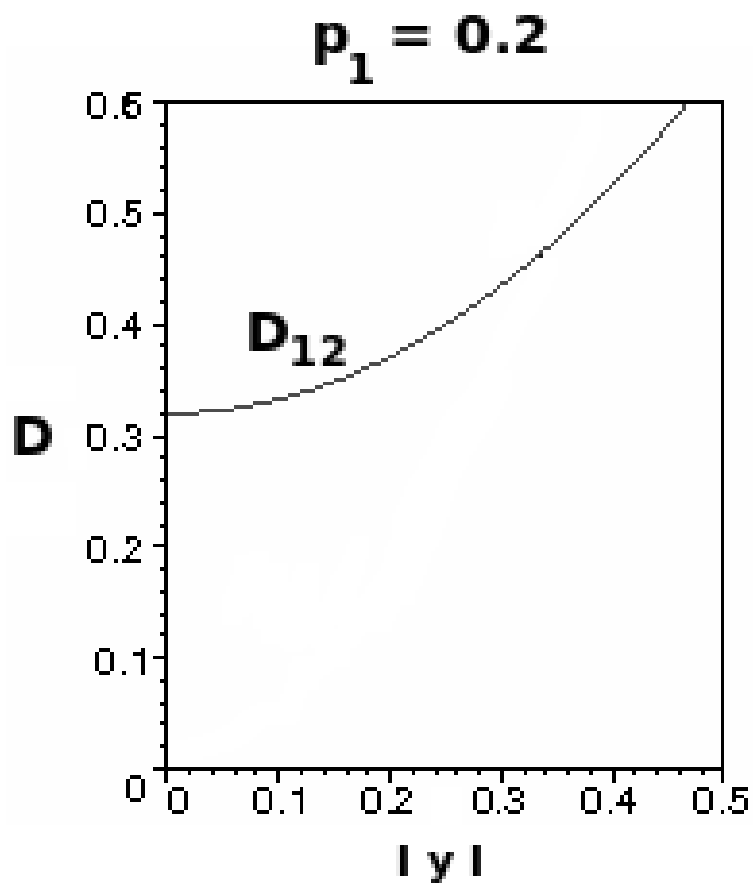[19] W. J. Munro, D. F. V. James, A. G. White, P. G. Kwiat, Phys. Rev. A **64**, 030302(2001).

Figure 1: The Hilbert Schmidt Norm D_{12} between the input state and the teleported state via quantum channel described by the density matrix of rank-2 is plotted versus the input state parameter $|y|$ when $p_1 = 0.2$
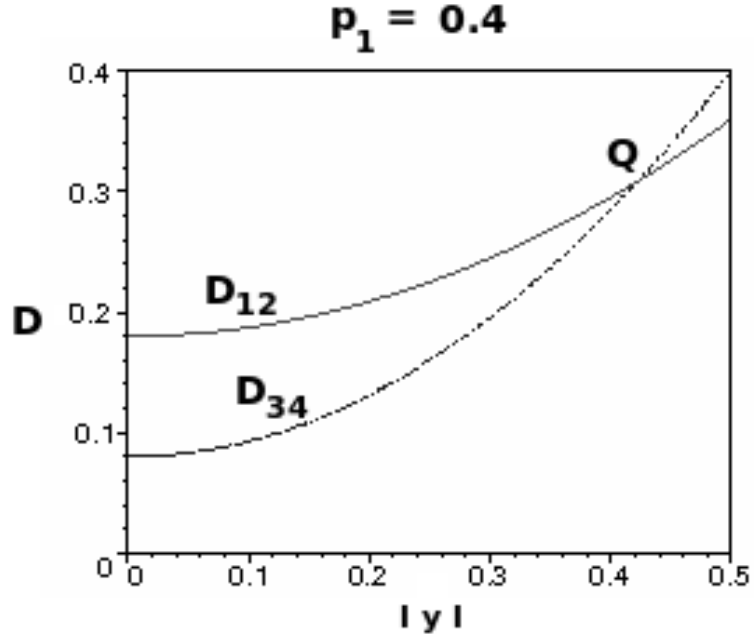
Figure 2: The Hilbert Schmidt Norms D_{12}, D_{34} between the input state and the teleported state via quantum channel described by the density matrix of rank-2 and rank-3 is plotted versus the input state parameter $|y|$ when $p_1 = 0.4$
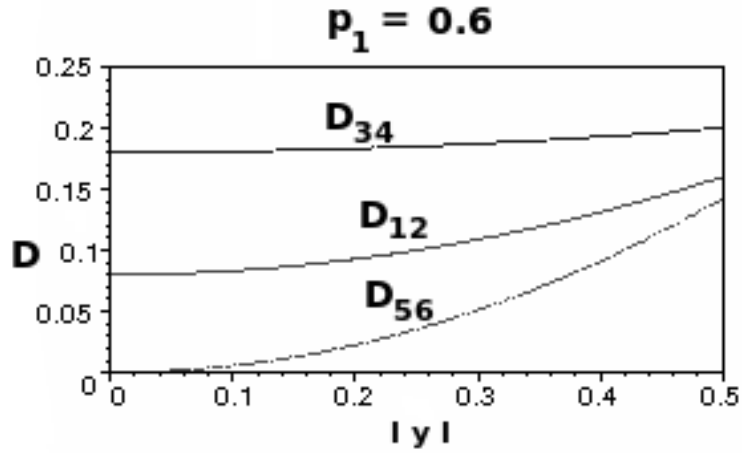

Figure 3: The Hilbert Schmidt Norms D_{12}, D_{34}, D_{56} between the input state and the teleported state via quantum channel described by the density matrix of rank-2, rank-3 and rank-4 is plotted versus the input state parameter $|y|$ when $p_1 = 0.6$
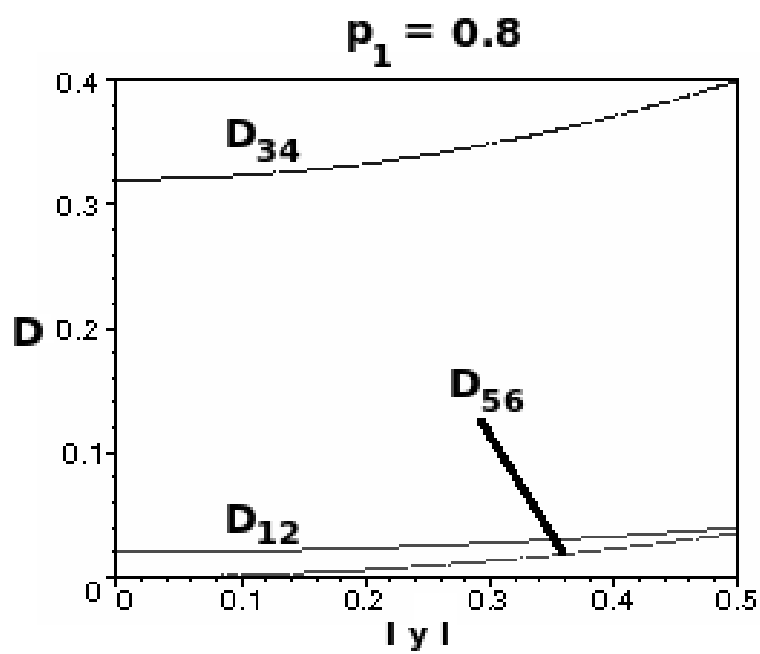
Figure 4: The Hilbert Schmidt Norms D_{12}, D_{34}, D_{56} between the input state and the teleported state via quantum channel described by the density matrix of rank-2, rank-3 and rank-4 is plotted versus the input state parameter $|y|$ when $p_1 = 0.8$
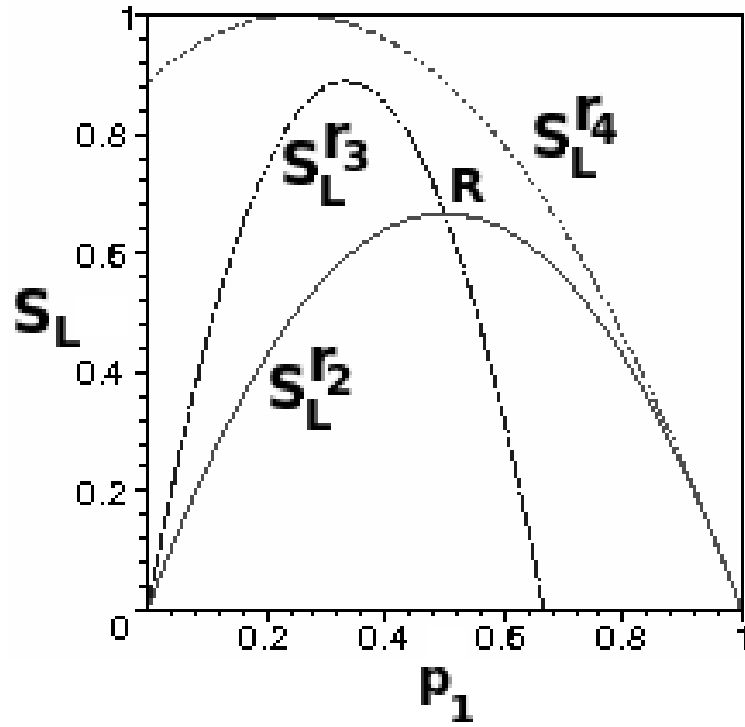
Figure 5: The linear entropies $S_L^{r_2}, S_L^{r_3}, S_L^{r_4}$ for two qubit teleportation channel described by the density matrix of rank-2, rank-3 and rank-4 are plotted versus the channel parameter p_1